Patrician College of Arts and Science

Department of Mathematics

Linear Algebra

TAM6A

Even Semester

Presented By
Mrs.T.Christy

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VECTOR SPACES AND SUBSPACES

- **Definition:** A **vector space** is a nonempty set *V* of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in *V* and for all scalars *c* and *d*.
  1. The sum of **u** and **v**, denoted by $\mathbf{u}+\mathbf{v}$, is in *V*.
  2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
  3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
  4. There is a zero vector 0 in *V* such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

5. For each **u** in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of **u** by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

VECTOR SPACES AND SUBSPACES

- **Example 1**. The space of all 3x3 matrices is a vector space. The space of all matrices is **not** a vector space.

SUBSPACES

- **Definition:** A **subspace** of a vector space *V* is a subset *H* of *V* that has three properties:
  a. The zero vector of *V* is in *H*.
  b. *H* is closed under vector addition. That is, for each **u** and **v** in *H*, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c. *H* is closed under multiplication by scalars. That is, for each **u** in *H* and each scalar *c*, the vector $c\mathbf{u}$ is in *H*.

SUBSPACES

- Properties (a), (b), and (c) guarantee that a subspace H of V is itself a *vector space*, under the vector space operations already defined in V.

- Every subspace is a vector space.

- Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

- **Example 1**. The space of all 3x3 upper triangular matrices is a subspace. The space of all matrices with integer entries is not.

A SUBSPACE SPANNED BY A SET

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

- As the term **linear combination** refers to any sum of scalar multiples of vectors, and Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

A SUBSPACE SPANNED BY A SET

- **Example 2:** Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space *V*, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that *H* is a subspace of *V*.
- **Solution:** The zero vector is in *H*, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
- To show that *H* is closed under vector addition, take two arbitrary vectors in *H*, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \text{ and } \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

- By Axioms 2, 3, and 8 for the vector space *V*,

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

A SUBSPACE SPANNED BY A SET

- **Theorem 1:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

- We call Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

- Give any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in H such that

$$H = \text{Span}\{\text{v}_1,\ldots\text{v}_p\}.$$

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- **Definition:** A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that
  i. $T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all **u**, **v** in V, and
  ii. $T(c\mathrm{u}) = cT(\mathrm{u})$ for all **u** in V and all scalars c.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- The **kernel** (or **null space**) of such a T is the set of all **u** in V such that $T(\mathbf{u}) = 0$ (the zero vector in W).
- The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some **x** in V.
- The kernel of T is a subspace of V.
- The range of T is a subspace of W.

Thank you

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